Session 3 : Fall 2016 Linear Algebra Writtens' Workshop

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1 Tips:

- Workshop complementary addition to the course. Problem Solving strategies.
- Good grasp of theory discussed in the LinAlg course and additional topics discussed in the workshop
- Preparation : Focus on problem solving - Time division - Topic wise study.
- Further discussion on the forum. Wiki is a good guide.
- Discussing tough problem in study circles(if that works for you)
- Supplementary resources and problems on the portal
- Peter Lax's book : Great Resource

2 Problems

2.1 Discrete Fourier Transform and Circulant matrices

January 2008 Problem 1

Let $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, $a, b, c > 0$. Find eigenvalues and a basis of eigenvectors of the matrix A.

2.2 Special matrices

September 2011 Problem 5

Consider a real $n \times n$ matrix where all the elements are zero except on the diagonal and those in the first row and first column. Also assume that all diagonal elements differ from zero and that all elements in the first row and first column are strictly positive.

Part 1
Prove that all the eigenvalues of such matrix are real.

Part 2
Prove that the rank of such a matrix is either n or $n - 1$

Follow-up : January 2006 Problem 5

Consider a tridiagonal $n \times n$ matrix A with real elements,

$$A = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \cdots & \\ c_1 & a_2 & b_2 & 0 & \cdots & \\ 0 & c_2 & a_3 & b_3 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & b_{n-1} \\ 0 & 0 & 0 & 0 & c_{n-1} & a_n \end{bmatrix}$$

Note that all elements $a_{ij} = 0$ for $|i - j| > 1$
Assume that the off-diagonal elements b_i and c_i are all strictly positive.

Part 1
Show that any such matrix has rank $n - 1$ or n

Part 2
Show that all eigenvalues are real.

2.3 Gershgorin Theorem and Stochastic Matrices

January 2006 Problem 4

The elements of a square stochastic matrix S are all nonnegative. Furthermore, the sum of the elements in each column is equal to 1.

Part 1
Show that $\lambda = 1$ is an eigenvalue of S.

Part 2
Show that all eigenvalues of S lie in the closed unit disc.

2.4 An interesting integral

(a) Compute

$$\int\int_{\mathbb{R}^2} \frac{dx\,dy}{(1+4x^2+9y^2)^2}, \quad \int\int_{\mathbb{R}^2} \frac{dx\,dy}{(1+5x^2-4xy+5y^2)^2}$$

(b) More Generally, Given $h(t)$ a given function such that $\int_0^\infty h(s)ds = \alpha$ and C and positive definite 2×2 matrix, show that

$$\int\int_{\mathbb{R}^2} h(< x, Cx >)\ dA = \frac{\pi\alpha}{\sqrt{det\ C}}$$

(c) Try at home :

$$\int\int_{\mathbb{R}^n} e^{-[<x,Ax>]} dx = \frac{\pi^{n/2}}{\sqrt{det\ A}}$$

for A positive definite $\in \mathbb{R}^{n \times n}$

(d) Try at home :

$$\int\int_{\mathbb{R}^n} e^{-[<x,Ax>+<b,x>+c]} dx = \frac{\pi^{n/2}}{\sqrt{det\ A}} e^{<b,A^{-1}b>-c}$$

(e) For a Symmetric matrix S, show that :

$$\int\int_{\mathbb{R}^n} < x, Sx > e^{-||x||^2} dx = \frac{1}{2}\pi^{n/2} Trace(S)$$

2.5 Matrix Exponential

January 2010 Problem 5

Let A be a $n \times n$ complex matrix, denote

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

Part 1
Let B be a $n \times n$ matrix, prove that $B = e^A$ for some matrix $A \iff det(B) \neq 0$.

Part 2
Let

$$B = \begin{bmatrix} 8/3 & -1/3 & -1/3 \\ 1/6 & 8/3 & -5/6 \\ 2/3 & -1/3 & 5/3 \end{bmatrix}$$

find an A such that $e^A = B$

2.6 Perturbation series

September 2012 Problem 5

Let A be a real, symmetric $n \times n$ matrix with n distinct, non-zero eigenvalues, and let V be a real, symmetric $n \times n$ matrix. Consider $A_\epsilon = A + \epsilon V$ a perturbation of A, where ϵ is a small number.

Part 1
Find an expression as an ϵ-series for the eigenvalues and eigenvectors of A_ϵ around the eigenvalues and eigenvectors of the original matrix A

Part 2
Assume the ϵ is sufficiently small that one can neglect all terms of order ϵ^2 and higher in the expansion. How close are the eigenvalues of A_ϵ to those of A? What about the eigenvectors?

Part 3
What can you say about the eigenvalues and eigenvectors of A_ϵ if A and V are arbitrary $n \times n$ matrices? You may assume that $det\ A \neq 0$

2.7 Matrix Calculus

(a) Derivative of products :

$$\frac{d}{dt}A(t)B(t) = \Big[\frac{d}{dt}A\Big]B + A\Big[\frac{d}{dt}B\Big]$$

(b) Let A be a matrix valued function, differentiable and invertible. Then A^{-1} is differentiable and

$$\frac{d}{dt}A^{-1} = -A^{-1}\Big[\frac{d}{dt}A\Big]A^{-1}$$

(c) Try at home : Relation between Determinant and Trace

$$\frac{d}{dt}log\ det\ A = Tr\Big(A^{-1}\frac{d}{dt}A\Big)$$

2.8 Affine transformation

January 2016 Question 5

5. Introduction: The surface $L = \{\vec{x} \in \mathbb{R}^3 \mid \|\vec{x}\|_1 = 1\}$ is a regular octohedron, and the surface $C = \{\vec{x} \in \mathbb{R}^3 \mid \|\vec{x}\|_\infty = 1\}$ is a regular cube. You might spend a few moments thinking about the tangent-planes to these surfaces, as well as imagining various ellipsoids centered at the origin, both circumscribing and circumscribed by regular octohedrons and cubes. Now consider an invertible linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$. Let S^2 be the unit-sphere in $\mathbb{R}^3$, and define the following vectors:

$$\begin{aligned}
\vec{v}_1 &= \arg\max_{\vec{x}\in S^2} \|A\vec{x}\|_1\,,\ \vec{w}_1 = \arg\min_{\vec{x}\in S^2} \|A\vec{x}\|_1\,.\\
\vec{v}_2 &= \arg\max_{\vec{x}\in S^2} \|A\vec{x}\|_2\,,\ \vec{w}_2 = \arg\min_{\vec{x}\in S^2} \|A\vec{x}\|_2\,.\\
\vec{v}_\infty &= \arg\max_{\vec{x}\in S^2} \|A\vec{x}\|_\infty\,,\ \vec{w}_\infty = \arg\min_{\vec{x}\in S^2} \|A\vec{x}\|_\infty\,.
\end{aligned}$$

as well as the sets:

$$\begin{aligned}
N_1 &= \{\vec{x} \in \mathbb{R}^3 \mid |\vec{x}_1| = |\vec{x}_2| = |\vec{x}_3| = 1\}\,,\\
N_\infty &= \{\vec{x} \in \mathbb{R}^3 \mid \text{exactly 2 components of } \vec{x} \text{ are } 0\}\,.
\end{aligned}$$

(a) Show that there exists a vector $\vec{n}_1 \in N_1$ such that, for all $\vec{x}_1 \perp \vec{v}_1$, we have $A\vec{x}_1 \perp \vec{n}_1$. Similarly, show that there exists a vector $\vec{n}_\infty \in N_\infty$ such that, for all $\vec{x}_\infty \perp \vec{v}_\infty$, we have $A\vec{x}_\infty \perp \vec{n}_\infty$.

(b) Show that $A\vec{w}_1$ can be chosen parallel to a vector in N_∞, and that $A\vec{w}_\infty$ can be chosen parallel to a vector in N_1.

(c) Find lower bounds for the absolute-value of the cosine of the angle between $A\vec{v}_1$ and $A\vec{v}_2$. Do the same for the absolute-value of the cosine of the angle between $A\vec{w}_1$ and $A\vec{w}_2$.

(d) Now let's assume that we are calculating the matrix-vector product $A\vec{x} = \vec{b}$, with $\vec{x} \neq 0$. Let's perturb x by a small ε in the direction $\Delta\vec{x}$, inducing a small change $\Delta\vec{b} = \varepsilon A\Delta\vec{x}$. Define the 'condition numbers' κ_1 and κ_∞ to be:

$$\kappa_1(x, \Delta x) = \lim_{\varepsilon\to 0} \frac{\left|\Delta\vec{b}\right|_1}{|A\vec{x}|_1} \cdot \frac{|\vec{x}|_2}{|\varepsilon\Delta\vec{x}|_2}, \text{ and } \kappa_\infty(x, \Delta x) = \lim_{\varepsilon\to 0} \frac{\left|\Delta\vec{b}\right|_\infty}{|A\vec{x}|_\infty} \cdot \frac{|\vec{x}|_2}{|\varepsilon\Delta\vec{x}|_2}.$$

For which directions $\vec{x}, \Delta\vec{x}$ is κ_1 as large as possible? For which directions $\vec{x}, \Delta\vec{x}$ is κ_∞ as large as possible?

September 2015 Question 4

4. Let $P \subset \mathbb{R}^n$ be the affine subspace defined by

$$\sum_{k=1}^{n} x_k = 1\,.$$

Let $e_n \in P$ be the standard basis elements of $\mathbb{R}^n$ along the coordinate directions. Let $T \subset P$ be the convex hull of the points $e_1, \ldots, e_n$ in P. The euclidean distance is $\|x - y\|$, and $B_r(x) \subset P$ is the ball of radius r in P centered about x. Find the radius of the smallest ball in P that contains T and the radius of the largest ball in P that is contained in T.

September 2005 Question 4

A simplex $T \subseteq \mathbb{R}^n$ is a set defined by $n+1$ inequalities: $x \in T \Leftrightarrow \langle u_k, x\rangle < c_k$ for $k = 1, \ldots, n+1$. An affine transformation is a map $\mathbb{R}^n \to \mathbb{R}^n$ defined by $x \to Ax + b$. The standard simplex $S \subseteq \mathbb{R}^n$ is the set defined by the inequalities $x_j > 0$ for $j = 1, \ldots, n$ and $\sum_{j=1}^n x_j < 1$. Show that any nonempty bounded simplex may be mapped to the standard simplex by a one to one affine transformation.

Hint: Define v_k by $\langle u_j, v_k\rangle = c_j$ for all $j \neq k$. What is the geometric interpretation of the v_k? Show that if T is bounded then each of the v_k is in the closure of T, which is to day $\langle u_k, v_k\rangle < c_k$.