Session 2 : Fall 2016 Linear Algebra Writtens' Workshop

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1 Problems

1.1 Determinants, Trace and Rank

January 1995 Problem 2

Suppose A is an $n \times n$ matrix and that

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, x = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, y^* = (y_1..y_n)$$

Suppose that all the entries of A, x and y are real.

Part 1
Show that there exist number a and b so that $det(A + sxy^*) = a + bs$.

Part 2
Show that if $det(A) \neq 0$ then $a = det(A)$ and $b = det(A)y^* A^{-1} x$.

Part 3
Is it true that $a = 0$ if $det(A) = 0$?

January 2012 Problem 2

Let $H = (h_{ij})$ be an $n \times n$ matrix such that (i) All coefficients $h_{ij} \in \{+1, -1\}$ and (ii) The row vectors $h_i = (h_{i1}, ..., h_{in})$ are mutually orthogonal

Part 1
Find a simple expression for HH^T

Part 2
Find $det(H)$.

Part 3
Let $u = (u_1, ..., u_n)$ with all $u_j = \pm 1$. Prove that at least one coordinate of Hu^T has absolute value at least $\sqrt{n}$. (Hint : Find the Euclidean norm of Hu^T).

January 2014 Problem 1

Consider $n \times n$ real matrices A, B and C with $ABC = 0$.

- What can be the maximal possible rank of CBA?
- What is a maximal possible rank of CBA is we assume that C, A are diagonal?
- What is a maximal possible rank of CBA is we assume that A, B, C are symmetric?

1.2 Projection and Orthogonal Matrix

January 2014 Problem 1

A complex $n \times n$ matrix U is unitary if it satisfies $U^H U = I$. A real unitary matrix is called orthogonal.

1. If U is $n \times n$ unitary, show that
$$max_{j,k}|U_{j,k}| \geq \frac{1}{\sqrt{n}}$$
with equality if and only if all of the elements of U are equal in magnitude.

2. Give an example of a 3×3 unitary matrix U for which all of the elements of U are equal in magnitude.

3. Prove the existence for every positive integer n of an $n \times n$ unitary matrix U for which all of the elements of U are equal in magnitude.

4. Now consider orthogonal matrices. For which $n \in \{2, 3, 4, 5, 6\}$ do there exist orthogonal matrices with all of the (real) elements equal in magnitude? For those $n \in \{2, 3, 4, 5, 6\}$ for which such matrices, exist, construct one; and for the remaining $n \in \{2, 3, 4, 5, 6\}$, prove that no such orthogonal matrix exists.

5. Find an infinite set S of positive integers such that for every $n \in S$ there exists an $n \times n$ orthogonal matrix all of whose elements are equal in magnitude.

September 2012 Problem 2

Find 3×3 matrices A, B and C that correspond to the following three linear operations. In all steps, explain your answers.

Part 1
Let A represent the orthogonal projection onto the plane $x - y + z = 0$.

Part 2
Let B represent the reflection across the plane $x - y + z = 0$.

Part 3
Let C represent rotation by π about the line $x = -y = z$. (Fortunately, it doesn't matter the sign of rotation, as either direction is the same.)

Part 4
Evaluate A^4, B^2 and C^3.

Part 5
Evaluate $AB - BA$, $AC - CA$, and $BC - CB$.

September 2015 Problem 3

In this problem, $\mathcal{M}_n(\mathbb{C})$ is the set of square matrices with size $n \times n$ with coefficients in $\mathbb{C}$, and for any matrix A, A^* is its conjugate transpose, defined such that $A^*_{ij} = \bar{A}_{ji}$, $\mathcal{U}_n(\mathbb{C})$ is the set of unitary matrices in $\mathcal{M}_n(\mathbb{C})$, i.e. matrices U whose adjoints U^* are also their inverses: $U^*U = UU^* = I_n$
Let k be an integer in $[1, n]$. We consider a matrix $P \in \mathcal{M}_n(\mathbb{C})$ that satisfies the conditions $(\mathcal{P}_k)$ given by :

$$(\mathcal{P}_k) \qquad P^2 = P = P^*, \quad rk(P) = k$$

1. Show that the entries of P satisfy
$$\forall i \in \mathbb{N} \; s.t. \; 1 \leq i \leq n, 0 \leq P_{ii} \leq 1,$$

$$\sum_{i=1}^{n} P_{ii} = k$$

2. Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be n real numbers and D the diagonal matrix such that $D_{ii} = \lambda_i$ for all integers i such that $1 \leq i \leq n$. Show that for any matrix P satisfying $(\mathcal{P}_k)$

$$Trace(PD) \leq \sum_{i=1}^{k} \lambda_i$$

Find a matrix Q that satisfies $(\mathcal{P}_k)$ and such that $Trace(PD) = \sum_{i=1}^{k} \lambda_i$

3. Show that if P_1 and P_2 are two matrices satisfying $(\mathcal{P}_k)$, there exists $U \in \mathcal{U}_n(\mathbb{C})$ such that $P_2 = UP_1U^*$. Show then that

$$\sum_{i=1}^{k} \lambda_i = max_{U \in \mathcal{U}_n(\mathbb{C})} \ Trace(UPU^*D)$$

where P is a matrix satisfying $(\mathcal{P}_k)$.

1.3 Inner Product and Norms

January 2016 Problem 4

Introduction : you might recall the double-angle formulas $2sin^2\theta = (1 - cos2\theta)$ and $2cos^2\theta = (1 + cos2\theta)$. These formulae, along with the change-of-variables $x = sin\theta$, can be used to show :

$$\int_{-1}^{+1} \frac{1}{\sqrt{1-x^2}} dx = \pi, \ and \quad \int_{-1}^{+1} \frac{x^2}{\sqrt{1-x^2}} dx = \pi/2$$

$$\int_{-1}^{+1} \frac{x^4}{\sqrt{1-x^2}} dx = 3\pi/8, \ and \quad \int_{-1}^{+1} \frac{x^6}{\sqrt{1-x^2}} dx = 5\pi/16$$

Now consider the vector-space P_n of polynomials with real coefficients of degree n or less defined on the interval $x \in [-1, 1]$. Endow this space with the following inner product :

$$< f, g >_T = \int_{-1}^{+1} f(x)g(x) \frac{dx}{\sqrt{1-x^2}}$$

(a) Find a basis $\{t_0(x), t_1(x), t_2(x)\}$ for P_2 that is orthonormal with respect to $< ., . >_T$ where t_0 is constant, $t_1(x)$ is linear, and $t_2(x)$ is quadratic.

(b) Express the integration operator $\int_{-1}^{+1} (.) dx$ as a linear operator on P_2 in terms of the basis t_0, t_1, t_2.

(c) Now consider the polynomial

$$a(x) = \sqrt{\frac{2}{\pi}} \Big[4x^3 + 2x^2 - 3x + x - 1 - \frac{1}{\sqrt{2}} \Big] \in P_3$$

Find the polynomial $b(x) \in P_2$ that is 'closest' to $a(x)$ in the $T-$norm. That is, find the $b(x)$ that minimizes :

$$< a - b, a - b >_T = \int_{-1}^{+1} (a(x) - b(x))^2 \frac{dx}{\sqrt{1-x^2}}.$$

(d) Now, with n arbitary, find a map from P_n to P_2 that sends a polynomial $a(x) \in P_n$ to the polynomial $b(x) \in P_2$ which is closest to $a(x)$ in the $T-$norm.

1.4 Jordan Form

January 2014 Problem 3

Find $lim_{N\to\infty}||A^N(x)||/||B^N(x)||$ as a function of $x \in \mathbb{R}^2$ if the matrices A, B are :

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}, \; B = \begin{bmatrix} 6 & -2 \\ 2 & 2 \end{bmatrix}$$

1.5 Related Problems for practice

S97Q5, J98Q2, J99Q3, J00Q4, S00Q2, J01Q5, S03Q1, S03Q2, J03Q5, S04Q5, J05Q1, J05Q2-Q3, J06Q3, J07Q2, J08Q3, J08Q5, J09Q3, S10Q1, S11Q3, S12Q2, S12Q4