Session 1 : Fall 2016 Linear Algebra Writtens' Workshop

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1 Problems

1.1 Spectral Theorem

September 2013 Problem 5

Let A be an Hermitean square matrix. Show that

$$rank(A) \geq \frac{[tr(A)]^2}{tr(A^2)}$$

1.2 Symmetric and Positive definite matrices

September 2014 Problem 1

A real $n \times n$ matrix A, not necessarily symmetric, is said to be positive definite if $x^T Ax > 0$ for every nonzero $x \in \mathbb{R}^n$

1. Show that a real $n \times n$ matrix A is positive definite is and only if the symmetric matric $(A^T + A)$ is positive definite.

2. If a real $n \times n$ matrix A is positive definite, show that $(I + \lambda A)$ is invertible for every $\lambda \geq 0$

3. Consider the system of differential equations

$$\frac{dx}{dt} = -Ax$$

   in which $x(t) \in \mathbb{R}^n$ for each t, and $A(t)$ is real $n \times n$ matrix for each t, with $A(t)$ positive definite, but not necessarily symmetric. Show that

$$\frac{d}{dt}(x^T x) \leq 0,$$

   at each t, which is equality if and only if $x(t) = 0$

4. Consider the linear system

$$x - y = -A(x + y)$$

   where $x \in \mathbb{R}^n$ is unknown, $y \in \mathbb{R}^n$ is given; and where A is real $n \times n$, and positive definite, but not necessarily symmetric.

   Prove the existence and uniqueness of x satisfying the above equation, and moreover that

$$x^T x \leq y^T y$$

   with equality if and only if $y = 0$

September 2013 Problem 3

Consider the square matrix $A = [a_{ij}] \in M_n$ with

$$a_{ij} = min(i, j).$$

Prove that A is positive definite.

September 2008 Problem 3

Part 1 : Find a lower triangular matrix, L, so that $LL^T = A$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 5 & 2 & 2 \\ 0 & 2 & 2 & 1 \\ 2 & 2 & 1 & 6 \end{bmatrix}$$

Part 2 : Find the volume in $\mathbb{R}^4$ of the set of x with $x^T Ax \leq 1$. You may use the fact that the volume in $\mathbb{R}^4$ of the set of x with $|x|^2 = x^T x \leq 1$ is $\frac{1}{2}\pi^2$. Hint : this is not a calculus problem.

September 2010 Problem 1

Let

$$\rho(x, y) = \frac{1}{Z} exp\Big(-(x \;\; y).C.(x \;\; y)^T\Big).$$

where $C \in \mathbb{R}^{2\times 2}$ is a positive definite symmetric matrix and $Z > 0$ is a constant.

Part 1 :
Describe how to construct a unitary 2×2 matrix U such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} zw\rho(x, y)dxdy = 0,$$

where z and w are defined via

$$U.(z, w)^T = (x, y)^T$$

Part 2 :
Use these newfound coordinates z and w to find the value of Z such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x, y)dxdy = 1.$$

Hint : In case you forgot, $\int_{-\infty}^{\infty} e^{-a\nu^2} d\nu = \sqrt{\pi/a}$

1.3 Quadratic Forms

January 2010 Problem 1

Identify the surface

$$3x^2 + 5y^2 + 3z^2 + 2xy + 2xz + 2yz = 1$$

as an ellipsoid in $\mathbb{R}^3$. Find the principal directions and the radii.

September 2015 Problem 5

Let n be an integer greater than 1, α be a real number, and consider the quadratic form Q_α given by

$$\forall (x_1, ..., x_n) \in \mathbb{R}^n, Q_\alpha(x_1, ..., x_n) = \sum_{i=1}^{n} x_i^2 - \alpha(\sum_{i=1}^{n} x_i)^2$$

Find all the eigenvalues of Q_α in terms of α and n. What is the condition on α for Q_α to be positive definite?

September 2008 Problem 4

Suppose $F(A)$ is a quadratic function of a real symmetric matrix A. This means that there are numbers f_{ijkl} so that $F(A) = \sum_{i,j,k,k,l} f_{ijkl} a_{ij} a_{kl}$. Suppose that $F(A) = F(QAQ^T)$ for every orthogonal matrix Q. Show that there are numbers c and d so that $F(A) = cTr(A^2) + d(Tr(A))^2$. Here $Tr(A)$ is the trace of A.

1.4 Matrix Equations

September 2011 Problem 4

Consider the problem

$$AX + XA = B$$

where A,X, and B are real $n \times n$ matrices. Here A and B are given, but X is unknown. Let A be symmetric and positive definite. That is, $A^T = A$ and $z^T Az > 0$ for all nonzero $z \in \mathbb{R}^n$.

1. Is this a linear system of equations? If so, how many equations and how many unknowns does it have?
2. Prove that the solution exists and is unique.
3. Solve for X. (Hint : Diagonalize A)

September 1998 Problem 2

Let A and B be square matrices that do not commute, i.e. such that $AB \neq BA$. Suppose further that A, B, and $(A + B)$ are each invertible. Show that $A(A + B)^{-1}B = B(A + B)^{-1}A$

September 2009 Problem 4

Consider the problem

$$(A + uv^T)x = b$$

which is to be solved for x given A, u, v, b where A is a non-singular $n \times n$ matrix and u, v, b are each $n \times 1$, as is the unknown vector x.

Part 1 :

Look for a solution of the form

$$x = A^{-1}b + \lambda A^{-1}u$$

where λ is a scalar. Solve for λ in terms of the given data.

Part 2 :

Let d be the denominator in your formula for λ. Show that

$$(d = 0) \implies (det(A + uv^T) = 0)$$

- Other Matrix Equations
- Sherman Morisson formula

1.5 Related Problems for practice

J95Q4, S96Q5, S98Q4, J01Q4(Courant Minmax theorem), S01Q4, S02Q3, S02Q5, S09Q3, J10Q3, J10Q4, S10Q2, S14Q4,